Comment on "New ansatz for metric operator calculation in pseudo-Hermitian field theory"

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In a recent Brief Report by Shalaby a new first-order perturbative calculation of the metric operator for an $i\phi^3$ scalar field theory is given. It is claimed that the result is an improvement on a previous calculation by Bender, Brody and Jones because it is local. Unfortunately Shalaby's calculation is not valid because of sign errors.

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The possibility that a non-Dirac-Hermitian $\mathcal{PT}$-symmetric Hamiltonian, such as $H = p^2 + ix^3$, could define a consistent quantum-mechanical theory was first pointed out in Ref. [1]. It was subsequently shown [2, 3, 4] that such Hamiltonians are self-adjoint with respect to the $\mathcal{CPT}$ operator, where $\mathcal{C}$ is a linear operator satisfying the following system of three simultaneous algebraic equations:

$$\mathcal{C}^2 = 1, \qquad [\mathcal{C}, \mathcal{PT}] = 0, \qquad [\mathcal{C}, H] = 0. \tag{1}$$

For cubic Hamiltonians of the general form $H = H_0 + \epsilon H_1$, where in quantum mechanics

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}x^2 + i\epsilon x^3 \tag{2}$$

and in quantum field theory

$$H = \int d^D x \left[\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2 + i\epsilon\phi^3\right], \tag{3}$$

it is explained in Refs. [3, 4] how to calculate the $\mathcal{C}$ operator using perturbative methods. The procedure is to take the $\mathcal{C}$ operator to have the form

$$\mathcal{C} = e^{\mathcal{Q}}\mathcal{P}, \tag{4}$$

where $\mathcal{Q}$ is a Hermitian operator that has a formal power series in odd powers of ϵ: $Q = \epsilon\mathcal{Q}_1 + \epsilon^3\mathcal{Q}_3 + \epsilon^5\mathcal{Q}_5 + \cdots$. Substituting (4) into (1) and collecting powers of ϵ, one obtains a sequence of algebraic equations that can be solved successively to determine the coefficients $\mathcal{Q}_1$, $\mathcal{Q}_3$, $\mathcal{Q}_5$, and so on. The first three of these equations are

$$\begin{aligned}
[\mathcal{Q}_1, H_0] &= 2H_1, \\
[\mathcal{Q}_3, H_0] &= \tfrac{1}{6}[\mathcal{Q}_1, [\mathcal{Q}_1, H_1]], \\
[\mathcal{Q}_5, H_0] &= -\tfrac{1}{360}[\mathcal{Q}_1, [\mathcal{Q}_1, [\mathcal{Q}_1, [\mathcal{Q}_1, H_1]]]] + \tfrac{1}{6}[\mathcal{Q}_1, [\mathcal{Q}_3, H_1]] + \tfrac{1}{6}[\mathcal{Q}_3, [\mathcal{Q}_1, H_1]].
\end{aligned} \tag{5}$$

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In Ref. [4], Bender, Brody, and Jones calculated $\mathcal{Q}_1$ for a cubic quantum field theory in $D+1$-dimensional space-time defined by (3). They found an expression for $\mathcal{Q}_1$ as a nonlocal function of the field operator ϕ. We emphasize that the result for $\mathcal{Q}_1$ in Ref. [4] is correct.

Recently, Shalaby reported a new calculation of $\mathcal{Q}_1$ in Ref. [5], in which he obtained a different result from that in given in Ref. [4]. It is not surprising that there might be another solution for $\mathcal{Q}_1$ because, as is shown in Ref. [6], the solution to the algebraic equations (1) is not unique. Shalaby's reported solution is claimed to be *local* and this new solution is characterized as being less "cumbersome" than the earlier nonlocal result of Ref. [4].

Unfortunately, the calculation reported by Shalaby in Ref. [5] is wrong. The first error may be found in the fifth unnumbered equation after Eq. (4). This equation is said to contain a total derivative that integrates to zero, but the expression is only a total derivative when the space-time dimension is 2. The correct form of the integrand is in fact $\nabla^2\phi(x)[\nabla\phi(x)]^2$, which in general is not a total derivative. The second error, which occurs in the fifth line of the unnumbered equation before Eq. (3), is an incorrect treatment of the derivative of a delta function. This leads to incorrect signs in two further equations. When the signs are corrected, the resulting equations for the coefficients C_1, C_2, and C_3 become inconsistent, thus invalidating the reported result.

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